The Newtonian limit of fourth-order gravity

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Abstract

The weak-field slow-motion limit of fourth-order gravity will be discussed.

Let us consider the gravitational theory defined by the Lagrangian

$$L_{\mathrm{g}} = (8\pi G)^{-1} \left(R/2 + (\alpha R_{ij} R^{ij} + \beta R^2) l^2 \right) . \tag{1}$$

G is Newton's constant, l a coupling length and α and β numerical parameters. R_{ij} and R are the Ricci tensor and its trace. Introducing the matter Lagrangian L_{m} and varying $L_{\mathrm{g}} + L_{\mathrm{m}}$ one obtains the field equation

$$E_{ij} + \alpha H_{ij} + \beta G_{ij} = 8\pi G T_{ij} . \tag{2}$$

For $\alpha = \beta = 0$ this reduces to General Relativity Theory. The explicit expressions H_{ij} and G_{ij} can be found in STELLE (1978).

In a well-defined sense, the weak-field slow-motion limit of Einstein's theory is just Newton's theory, cf. DAUTCOURT (1964). In the following we consider the analogous problem for fourth order gravity eqs. (1), (2). For the special cases $\alpha = 0$ (PECHLANER, SEXL(1966), POLIJEVKTOV-NIKOLADZE (1967)), $\alpha + 2\beta = 0$ (HAVAS (1977), JANKIEWICZ (1981))

and $\alpha + 3\beta = 0$ (BORZESZKOWSKI, TREDER, YOURGRAU (1978)) this has already been done in the past. Cf. also ANANDAN (1983), where torsion has been taken into account.

The slow-motion limit can be equivalently described as the limit $c \to \infty$, where c is the velocity of light. In this sense we have to take the limit $G \to 0$ while $G \cdot c$ and l remain constants. Then the energy-momentum tensor T_{ij} reduces to the rest mass density ρ:

$$T_{ij} = \delta_i^0 \delta_j^0 \rho \,, \tag{3}$$

$x^0 = t$ being the time coordinate. The metric can be written as

$$ds^2 = (1 - 2\phi) dt^2 - (1 + 2\psi)(dx^2 + dy^2 + dz^2) \,. \tag{4}$$

Now eqs. (3) and (4) will be inserted into eq. (2). In our approach, products and time derivatives of ϕ and ψ can be neglected, i.e.,

$$R = 4\Delta\psi - 2\Delta\phi \,, \qquad \text{where} \qquad \Delta f = f_{,xx} + f_{,yy} + f_{,zz} \,.$$

Further $R_{00} = -\Delta\phi$, $H_{00} = -2\Delta R_{00} - \Delta R$ and $G_{00} = -4\Delta R$, where $l = 1$.

Then it holds: The validity of the 00-component and of the trace of eq. (2),

$$R_{00} - R/2 + \alpha H_{00} + \beta G_{00} = 8\pi G \rho \tag{5}$$

and

$$-R - 4(\alpha + 3\beta)\Delta R = 8\pi G \rho \,, \tag{6}$$

imply the validity of the full eq. (2).

Now, let us discuss eqs. (5) and (6) in more details: Eq. (5) reads

$$-\Delta\phi - R/2 + \alpha(2\Delta\Delta\phi - \Delta R) - 4\beta\Delta R = 8\pi G \rho \,. \tag{7}$$

Subtracting one half of eq. (6) yields

$$-\Delta\phi + 2\alpha\Delta\Delta\phi + (\alpha + 2\beta)\Delta R = 4\pi G\rho\,. \tag{8}$$

For $\alpha + 2\beta = 0$ one obtains

$$-(1 - 2\alpha\Delta)\Delta\phi = 4\pi G\rho \tag{9}$$

and then $\psi = \phi$ is a solution of eqs. (5), (6). For all other cases the equations for ϕ and ψ do not decouple immediately, but, to get equations comparable with Poisson's equation we apply Δ to eq. (6) and continue as follows.

For $\alpha + 3\beta = 0$ one gets from eq. (8)

$$-(1 - 2\alpha\Delta)\Delta\phi = 4\pi G(1 + 2\alpha\Delta/3)\rho\,. \tag{10}$$

The Δ-operator applied to the source term in eq. (10) is only due to the application of Δ to the trace, the original equations (5), (6) contain only ρ itself.

For $\alpha = 0$ one obtains similarily the equation

$$-(1 + 12\beta\Delta)\Delta\phi = 4\pi G(1 + 16\beta\Delta)\rho\,. \tag{11}$$

For all other cases - just the cases not yet covered by the literature - the elimination of ψ from the system (5), (6) gives rise to a sixth-order equation

$$-(1 + 4(\alpha + 3\beta)\Delta)\,(1 - 2\alpha\Delta)\Delta\phi = 4\pi G\,(1 + 2(3\alpha + 8\beta)\Delta)\,\rho\,. \tag{12}$$

Fourth-order gravity is motivated by quantum-gravity considerations and therefore, its long-range behaviour should be the same as in Newton's theory. Therefore, the signs of the parameters α, β should be chosen to guarantee an exponentially vanishing and not an oscillating behaviour of the fourth-order terms:

$$\alpha \geq 0\,, \qquad \alpha + 3\beta \leq 0\,. \tag{13}$$

On the other hand, comparing parts of eq. (12) with the Proca equation it makes sense to define the masses

$$m_2 = (2\alpha)^{-1/2} \quad \text{and} \quad m_0 = (-4(\alpha + 3\beta))^{-1/2} \ . \tag{14}$$

Then (13) requires the masses of the spin 2 and spin 0 gravitons to be real.

Now, inserting a delta source $\rho = m\delta$ into eq. (12) one obtains for ϕ the same result as STELLE (1978),

$$\phi = mr^{-1} \left(1 + \exp(-m_0 r)/3 - 4\exp(-m_2 r)/3\right) \ . \tag{15}$$

To obtain the metric completely one has also to calculate ψ. It reads

$$\psi = mr^{-1} \left(1 - \exp(-m_0 r)/3 - 2\exp(-m_2 r)/3\right) \ . \tag{16}$$

For finite values m_0 and m_2 these are both bounded functions, also for $r \to 0$. In the limits $\alpha \to 0$ ($m_2 \to \infty$) and $\alpha + 3\beta \to 0$ ($m_0 \to \infty$) the terms with m_0 and m_2 in eqs. (15) and (16) simply vanish. For these cases ϕ and ψ become unbounded as $r \to 0$.

Inserting (15), (16) into the metric (4), the behaviour of the geodesics shall be studied. First, for an estimation of the sign of the gravitational force we take a test particle at rest and look whether it starts falling towards the centre or not. The result is: for $m_0 \le 2m_2$, gravitation is always attractive, and for $m_0 > 2m_2$ it is attractive for large but repelling for small distances. The intermediate case $m_0 = 2m_2$, i.e., $3\alpha + 8\beta = 0$, is already known to be a special one from eq. (12).

Next, let us study the perihelion advance for distorted circle-like orbits. Besides the general relativistic perihelion advance (which vanishes in the Newtonian limit) we have an additional one of the following behaviour: For $r \to 0$ and $r \to \infty$ it vanishes and for $r \approx 1/m_0$ and $r \approx 1/m_2$ it has local maxima, i.e., resonances.

Finally, it should be mentioned that the gravitational field of an extended body can be obtained by integrating eqs. (15), (16). For a spherically symmetric body the far field is also of the type

$$mr^{-1}\,(1 + a\exp(-m_0 r) + b\exp(-m_2 r))\ ,$$

and the factors a and b carry information about the mass distribution inside the body.

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